SYMBOLIC COMPUTATION OF NORMAL FORMS FOR RESONANT DOUBLE HOPF BIFURCATIONS USING A PERTURBATION TECHNIQUE

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This paper presents a perturbation method for computing the normal forms of resonant double Hopf bifurcations with the aid of a computer algebraic system. This technique, based on the method of multiple time scales, can be used to deal with general *n*-dimensional systems without the application of center manifold theory. Explicit iterative formulas have been derived for uniquely determining the coefficients of normal forms and associated non-linear transformations up to an arbitrary order. User-friendly symbolic computer programs written in Maple have been developed, which can be easily applied for computing the normal forms of a given system. Examples are presented to show the applicability of the methodology and the convenience of using the computer software.

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1. INTRODUCTION

Normal form theory has been widely used to simplify the analysis of dynamic behavior of differential equations near an equilibrium [1–3]. The basic idea of the method of normal forms is to employ successive co-ordinate transformations to systematically construct a simplified form of the original differential equations without changing the fundamental dynamical behavior of the system in the vicinity of the equilibrium. Normal form theory is usually applied along with center manifold theory [4]. Center manifold theory is applied before normal form theory, to obtain a locally invariant small dimensional manifold in the vicinity of an equilibrium, called center manifold. Although it is easy to construct an "abstract" normal form for a given singularity, it is difficult to compute the explicit expressions of a normal form in terms of the coefficients of the original differential equations. Therefore, the crucial part in computing the normal form of a system is the computation efficiency. Moreover, algebraic manipulations become very involved and time consuming as the order of the normal form increases. The introduction of computer algebra systems such as Maple, Mathematica, Reduce, etc., becomes necessary at this point. Recently, many researchers have paid attention to the development of efficient computational methods with the aid of computer algebra systems [5–11].

In this paper, a perturbation technique is used to compute the normal forms of differential equations whose Jacobian, evaluated at an equilibrium, has two pairs of purely imaginary eigenvalues. The ratio of the two eigenvalues is a rational number, giving rise to resonant oscillations. The perturbation method, combined with multiple time scales, has been successfully developed for computing normal forms of Hopf bifurcation and other singularities [7, 12, 13]. It has been shown that the approach is very computationally

efficient and is simple enough to use by a novice to computer algebra. This technique is systematic and can be directly applied to the original n-dimensional differential equations, without the application of centre manifold theory. This approach reduces the steps involved in the application of normal form theory to simultaneously obtain the normal form (on the center manifold) as well as the associated non-linear transformation.

This paper focuses on the development of the methodology and symbolic software for computing the normal forms of a double Hopf bifurcation associated with various resonant cases, including $k_1 : k_2$ ($k_1 \neq k_2$, k_1 and k_2 are positive integers) resonances and $1 : 1$ primary resonance. The separate treatment for the two cases is due to different scaling which is necessary for the application of perturbation techniques. Uniform scaling is used for the $k_1 : k_2$ ($k_1 \neq k_2$) resonant cases while non-uniform scaling has to be applied for the $1 : 1$ resonance to match the perturbation orders. All the formulas derived in this paper for computing the coefficients of the normal forms and the non-linear transformations are given explicitly in terms of the original system coefficients. This facilitates direct applications to special problems [8, 10, 14]. Based on the explicit recursive formulas, user-friendly symbolic software using Maple has been developed. Examples are presented to show that the perturbation technique is computationally efficient and therefore, is particularly useful for computing high-dimensional systems and high order normal forms.

The $k_1 : k_2$ ($k_1 \neq k_2$) resonant cases are described in the next section, while section 3 deals with $1 : 1$ resonant cases. Section 4 outlines the symbolic computer programs, and examples are presented in section 5. Conclusions are drawn in section 6.

2. PERTURBATION TECHNIQUE AND FORMULATIONS

2.1. $k_1 : k_2$ RESONANT CASES

Consider a general n-dimensional system described by the following differential equation:

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in R^n, \quad \mathbf{f} : R^n \to R_n, \tag{1}$$

where $J\mathbf{x}$ represents the linear term, and the non-linear function $\mathbf{f}$ is assumed to be analytic; and $\mathbf{x} = \mathbf{0}$ is an equilibrium of the system, i.e., $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Further, assume that the Jacobian of system (1) evaluated at the equilibrium $\mathbf{0}$ involves two pairs of purely imaginary eigenvalues $\pm i\omega_{1c}$ and $\pm i\omega_{2c}$. In this section, we assume that

$$\frac{\omega_{1c}}{\omega_{2c}} = \frac{k_1}{k_2}, \quad \text{where } k_1 \text{ and } k_2 \text{ are integers}, \tag{2}$$

which is called a $k_1 : k_2$ resonance. Thus, for the convenience of analysis, we may let $\omega_{1c} = k_1\omega$ and $\omega_{2c} = k_2\omega$, and further, without loss of generality, we may assume that $\omega = 1$. (Otherwise, one may use a time scaling $t' = \omega t$ to scale the frequency ω to 1.) Consequently, the Jacobian matrix of system (1) evaluated at the equilibrium, $\mathbf{x} = \mathbf{0}$, can be assumed (with the aid of linear transformation if necessary) in the Jordan canonical form, to be

$$J = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 \\ -k_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-4) \times (n-4)}, \tag{3}$$

such that $k_1 < k_2$. Here A is hyperbolic, i.e., all of its eigenvalues have non-zero real parts, with either a real or complex conjugate. For physically interesting cases, the unstable manifold is assumed to be empty, i.e., all the eigenvalues of A have negative real parts. Let us rewrite system (1) in component form as

$$\dot{x}_1 = k_1 x_2 + f_1(\mathbf{x}), \qquad \dot{x}_2 = -k_1 x_1 + f_2(\mathbf{x}), \tag{4}$$

$$\dot{x}_3 = k_2 x_4 + f_3(\mathbf{x}), \qquad \dot{x}_4 = -k_2 x_3 + f_4(\mathbf{x}), \tag{5}$$

$$\dot{x}_p = -\alpha_p x_p + f_p(\mathbf{x}) \quad (p = 5, 6, \ldots, m_1 + 4), \tag{6}$$

$$\dot{x}_q = -\alpha_q x_q + \omega_q x_{q+1} + f_q(x),$$

$$\dot{x}_{q+1} = -\omega_q x_q - \alpha_q x_{q+1} + f_{q+1}(\mathbf{x}) \quad (q = m_1 + 5, m_1 + 7, \ldots, n-1), \tag{7}$$

where $\alpha_p > 0$, $\alpha_q > 0$, $\omega_q > 0$, and $4 + m_1 + 2m_2 = n$. Equations (4)–(7) show that the matrix A has m_1 real eigenvalues, $-\alpha_p$, $p = 1, 2, \ldots, m_1$, and m_2 pairs of complex conjugate eigenvalues, $-\alpha_q \pm i\omega_q$, $q = 1, 2, \ldots, m_2$. The functions $f_i(\mathbf{x})$ satisfy $f_i(\mathbf{0}) = 0$ and $\partial f_i(\mathbf{0})/\partial x_j = 0$, $i, j = 1, 2, \ldots, n$. That is, the Taylor expansion of $f_i(\mathbf{x})$ about $\mathbf{x} = \mathbf{0}$ starts with quadratic terms.

The underlying idea of the method of multiple scales is to consider the expansion of representing the response as a function of multiple independent variables, or scales, instead of a single time variable. This can be achieved by introducing new independent variables according to

$$T_k = \varepsilon^k t, \quad k = 0, 1, 2, \ldots . \tag{8}$$

It follows that the derivatives with respect to t now become expansions in terms of the partial derivatives with respect to T_k, given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}T_0}{\mathrm{d}t}\frac{\partial}{\partial T_0} + \frac{\mathrm{d}T_1}{\mathrm{d}t}\frac{\partial}{\partial T_1} + \frac{\mathrm{d}T_2}{\mathrm{d}t}\frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \tag{9}$$

where the differentiation operator $D_k = \partial/\partial T_k$.

Now suppose that the solution of equation (1) (or equivalently, equations (4)–(7)) in the neighborhood of $\mathbf{x} = \mathbf{0}$ is represented by an expansion of the form

$$x_i(t; \varepsilon) = \varepsilon x_{i1}(T_0, T_1, \ldots) + \varepsilon^2 x_{i2}(T_0, T_1, \ldots) + \cdots \quad (i = 1, 2, \ldots, n). \tag{10}$$

Note that the number of independent time scales needed depends on the order to which the expansion is carried out. In general, to find a normal form up to an order n, $T_0, T_1, \ldots, T_n$ should be used in solution (10). However, one can employ the idea of the *intrinsic harmonic balance* (IHB) technique [15] and let n be open so that the process of the method will take into account all essential terms up to the required order, as seen in the following procedure.

Substituting solution (10) into equations (4)–(7) with the aid of equation (9) and balancing the like powers of ε results in the perturbation equations in the order given:

$$\varepsilon^1 : D_0 x_{11} = k_1 x_{21}, \qquad D_0 x_{21} = -k_1 x_{11}, \tag{11}$$

$$D_0 x_{31} = k_2 x_{41}, \qquad D_0 x_{41} = -k_2 x_{31}, \tag{12}$$

$$D_0 x_{p1} = -\alpha_p x_{p1} \quad (p = 3, 4, \ldots, m_1 + 2), \tag{13}$$

$$D_0 x_{q1} = -\alpha_q x_{q1} + \omega_q x_{(q+1)1},$$

$$D_0 x_{(q+1)1} = -\omega_q x_{q1} - \alpha_q x_{(q+1)1} \quad (q = m_1 + 3, \, m_1 + 5, \, \dots, \, n-1). \tag{14}$$

$$\varepsilon^2: \; D_0 x_{12} = k_1 x_{22} - D_1 x_{11} + f_{12}(\mathbf{x}_1), \qquad D_0 x_{22} = -k_1 x_{12} - D_1 x_{21} + f_{22}(\mathbf{x}_1), \tag{15}$$

$$D_0 x_{32} = k_2 x_{42} - D_1 x_{31} + f_{32}(\mathbf{x}_1), \qquad D_0 x_{42} = -k_2 x_{32} - D_1 x_{41} + f_{42}(\mathbf{x}_1), \tag{16}$$

$$D_0 x_{p2} = -\alpha_p x_{p2} + f_{p2}(\mathbf{x}_1) \quad (p = 3, \, 4, \, \dots, \, m_1 + 2), \tag{17}$$

$$D_0 x_{q2} = -\alpha_q x_{q2} + \omega_q x_{(q+1)2} + f_{q2}(\mathbf{x}_1),$$

$$D_0 x_{(q+1)2} = -\omega_q x_{q2} - \alpha_q x_{(q+1)2} + f_{(q+1)2}(\mathbf{x}_1) \quad (q = m_1 + 3, \, m_1 + 5, \, \dots, \, n-1), \tag{18}$$

etc., where $f_{i2} = (\mathrm{d}^2/\mathrm{d}\varepsilon^2) \left[f_i(\mathbf{x}_1) \right]_{\varepsilon=0}$ are functions of x_{i1} ($i = 1, 2, \dots, n$) which have been obtained from the ε^1 order perturbation equations (11)–(14). In general, functions f_{ik} only involve the variables which have been solved from the previous $1, 2, \dots, (k-1)$ order perturbation equations.

To find the solutions of the ε^1 order equations (11)–(14), first note that these equations can be divided into two groups, one of which consists of the first four equations given in equations (11) and (12), and the other one includes the remaining equations. The former group is associated with the critical eigenvalues (having zero real parts), and the latter corresponds to the non-critical eigenvalues (having non-zero real parts). Second, the first order solutions for the variables of the second group are obtained only from the first four variables x_1, x_2, x_3 and x_4, since the perturbation technique is based on the assumed asymptotic solution (10). However, it is interesting to note that the normal form obtained based on equation (10) not only represents the asymptotic behavior, but also the transient property of the system.

Now consider equation set (11). Differentiating the first equation of equation (11) and then substituting the second equation of equation (11) into the resulting equation yields a simple second order ordinary differential equation,

$$D_0^2 x_{11} + k_1^2 x_{11} = 0. \tag{19}$$

The solution of equation (19) can be written in a general form as

$$x_{11} = r_1(T_1, T_2, \dots) \cos \left[k_1 T_0 + \phi_1(T_1, T_2, \dots) \right] \equiv r_1 \cos(k_1 T_0 + \phi_1) \equiv r_1 \cos \theta_1, \tag{20}$$

where r_1 and ϕ_1 represent, respectively, the amplitude and phase of motion, and $\theta_1 = k_1 T_0 + \phi_1$. Once x_{11} is determined, x_{21} can be directly solved from the first equation of equation (11). Note that solution (20) implies that

$$D_0 r_1 = 0 \quad \text{and} \quad D_0 \phi_1 = 0, \tag{21}$$

since r_1 and ϕ_1 do not contain the variable T_0. (Remember that D_0 denotes differentiation with respect to T_0.) Similarly, we can obtain solutions from equation (14) as follows:

$$x_{31} = r_2(T_1, T_2, \dots) \cos \left[k_2 T_0 + \phi_2(T_1, T_2, \dots) \right] \equiv r_2 \cos(k_2 T_0 + \phi_1) \equiv r_2 \cos \theta_2 \tag{22}$$

and

$$D_0 r_2 = 0 \quad \text{and} \quad D_0 \phi_2 = 0. \tag{23}$$

x_{41} can be determined from the first equation of equation (12) after x_{31} is solved. The asymptotic ε^1 order solutions of the second group, described by equations (13) and (14), are obviously given by

$$x_{i1} = 0, \quad i = 5, 6, \dots, n, \tag{24}$$

which actually represent the first order steady state solutions of the second group equations.

Next, to solve the ε^2 order perturbation equations (15)–(18), the procedure described above can be applied. Thus, differentiating the first equation of equation (15) and then substituting the second equation of equation (15) into the resulting equation results in a non-linear homogeneous ordinary differential equation:

$$D_0^2 x_{12} + k_1^2 x_{12} = -D_1 D_0 x_{11} - D_1 x_{21} + D_0 f_{12} + f_{22}. \tag{25}$$

Then, substituting the solutions x_{11} and x_{21} into the right-hand side of equation (25) gives an expression in terms of the trigonometric functions $\cos k T_0$ and $\sin k T_0$. To eliminate possible *secular* terms which may appear in the solution of x_{12}, it is required that the coefficients of the two terms $\cos k_1 T_0$ and $\sin k_1 T_0$ be equal to zero, which in turn yields the explicit solutions for the derivatives $D_1 r_1$ and $D_1 \phi_1$. Then, the solution of x_{12} can be found from the remaining terms of equation (23), and thus, x_{12} contains only a particular solution. Having found x_{12}, x_{22} is determined from the first equation of equation (15) as $x_{22} = (1/k_1)(D_0 x_{12} + D_1 x_{11} - f_{12}(\mathbf{x}_1))$. This procedure can be carried out to higher order perturbation equations, and thus we can find $D_1 r_2$, $D_1 \phi_2$, x_{32} and x_{42}, etc. Finally, the normal forms, given in polar co-ordinates, can be written as

$$\frac{dr_1}{dt} = \frac{\partial r_1}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial r_1}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial r_1}{\partial T_2} \frac{\partial T_2}{\partial t} + \cdots = D_0 r_1 + D_1 r_1 + D_2 r_1 + \cdots, \tag{26}$$

$$\frac{d\theta_1}{dt} = k_1 + \frac{\partial \phi_1}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial \phi_1}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial \phi_1}{\partial T_2} \frac{\partial T_2}{\partial t} + \cdots = k_1 + D_0 \phi_1 + D_1 \phi_1 + D_2 \phi_1 + \cdots, \tag{27}$$

$$\frac{dr_2}{dt} = \frac{\partial r_2}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial r_2}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial r_2}{\partial T_2} \frac{\partial T_2}{\partial t} + \cdots = D_0 r_2 + D_1 r_2 + D_2 r_2 + \cdots, \tag{28}$$

$$\frac{d\theta_2}{dt} = k_2 + \frac{\partial \phi_2}{\partial T_0} \frac{\partial T_0}{\partial t} + \frac{\partial \phi_2}{\partial T_1} \frac{\partial T_1}{\partial t} + \frac{\partial \phi_2}{\partial T_2} \frac{\partial T_2}{\partial t} + \cdots = k_2 + D_0 \phi_2 + D_1 \phi_2 + D_2 \phi_2 + \cdots, \tag{29}$$

where the back scaling $\varepsilon r_i \to r_i$ (i.e., $\varepsilon x_i \to x_i$) has been used (or simply set $\varepsilon = 1$). It should be pointed out here that the particular solutions of equation (25), etc., can be found by using the IHB method [15] so that the solution is uniquely determined. Thus, $D_i r_j$ and $D_i \phi_j$ ($j = 1, 2$) are also uniquely defined, which implies that the normal form given in equations (26)–(29) are actually uniquely determined. It is also noted that, unlike the non-resonant case, in which the derivatives $D_i r_j$ and $D_i \phi_j$ are functions of r_1 and r_2 only, here they are given explicitly in terms of amplitudes r_1, r_2 and phase $k_2 \theta_1 - k_1 \theta_2$.

2.2. 1:1 RESONANT CASE

Again, consider system (1), but now the Jacobian evaluated on the equilibrium $\mathbf{x} = \mathbf{0}$ is given by

$$
J = \begin{bmatrix}
0 & \omega & 0 & 1 & 0 \\
-\omega & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \omega & 0 \\
0 & 0 & -\omega & 0 & 0 \\
0 & 0 & 0 & 0 & A
\end{bmatrix}, \quad A \in R^{(n-4)\times(n-4)}, \tag{30}
$$

where ω can be assumed as $\omega = 1$ if a time scaling $t' = \omega t$ is applied. Note that this is called the non-semisimple case since two 1's are present in J.

This resonant case is quite similar to the resonant cases discussed in section 2.1. The only difference for this particular case is that unlike a general $k_1 : k_2$ ($k_1 \neq k_2$) resonance, we cannot use a uniform scaling $x_i \rightarrow \varepsilon x_i$ to obtain the ordered perturbation equations. It is straightforward to show that a uniform scaling leads to the solution of non-linear differential equations at the first perturbation order. This difficulty is due to the non-semisimple resonance, that is, linear terms x_3 and x_4 are present in the first and second equations (see the Jacobian given by equation (30)). In order for the perturbation technique to work on the 1:1 non-semisimple resonant case, we must use different scalings between x_1, x_2 and other variables, x_i, $i = 3, 4, 5, \dots, n$. A careful consideration shows that the ratio of the perturbation order for x_1, x_2 and the remaining variables should be 2:3. Thus, let

$$
x_1 \rightarrow \varepsilon^2 x_1, \qquad x_2 \rightarrow \varepsilon^2 x_2, \qquad x_i \rightarrow \varepsilon^3 x_i, \quad i = 3, 4, \dots, n. \tag{31}
$$

Then, unlike the $k_1 : k_2$ ($k_1 \neq k_2$) resonances, where the highest degree of harmonics at an order k is consistently equal to $k + 1$, now the kth order terms in the original differential equations may spread into different order ($> k$) perturbation equations due to the different scalings. However, it can be shown that the highest order of the perturbation equation which should be included for the original kth order terms is $3k - 2$. Therefore, the highest degree of the harmonics for each kth order must be $3k - 2$. In other words, if an nth order normal form is needed, then the perturbation equations up to $3n - 2$ order must be considered. Once this scaling is applied and the solution form is appropriately set, then a solution procedure similar to that presented in section 2.1 can be obtained and the formulas similar to that given in section 2.1 can be found.

It should be noted, unlike for the $k_1 : k_2$ ($k_1 \neq k_2$) resonant cases, a kth order normal form and the associated non-linear transformations (solutions) obtained for this 1:1 case may involve higher order terms which actually do not belong to the kth order expression. This is due to the difference in scaling. This suggests that the difference in scaling has caused the lower order terms to spread into higher order perturbation equations, and has caused higher order terms to appear in lower order ($< 3n - 2$) perturbation equations. This is the fundamental difference between the 1:1 primary resonance and the $k_1 : k_2$ ($k_1 \neq k_2$) resonance. In order to remove the terms which do not belong to the nth order normal form, one may just easily truncate the normal form and solutions up to nth *order*. (Note: here the *order* means the order of the original differential equations.) This redundant calculation increases the complexity and the time required for the computation.

It is interesting to note that the normal form obtained using the perturbation technique via different scalings actually agree with that found by using other theories and methods

such as Poincaré normal form theory and the IHB technique. This will be demonstrated by the examples presented in section 4.

3. OUTLINE OF THE SYMBOLIC COMPUTATION

All the formulas presented in the previous section are given explicitly in terms of the coefficients of the original differential equations, and thus can be easily implemented using a computer algebraic system. Maple has been used to code these explicit formulas. In this section, we shall outline the Maple programs.[†]

3.1. MAPLE SOURCE CODE

(1) Read a prepared input file. The input file provides the original different equations, with the number of non-zero real eigenvalues, N_1, and the number of the pairs of complex conjugate eigenvalues, N_2, as well as the order for computing the normal form, *norder*.

(2) Create procedures.
   — The procedure *Findcoef* for computing all the coefficients of harmonics in terms of $\cos(m\theta_1 + n\theta_2)$ and $\sin(m\theta_1 + n\theta_2)$ for a given trigonometric function F.
   — The procedure *solu1A_B* for computing the coefficients of harmonics in solutions x_i, $i = 1, 2, 3, 4$ (corresponding to the center manifold) using the method of harmonic balancing.
   — The procedure *solu2A_B* for computing the coefficients of harmonics in solutions x_i, $i = 5, 6, \ldots, N_1 + 4$ (corresponding to the non-zero real eigenvalues) using the method of harmonic balancing.
   — The procedure *solu3A_B* for computing the coefficients of harmonics in solutions x_i, $i = N_i + 5, N_1 + 6, \ldots, N_1 + 2 \times N_2$ (corresponding to the complex conjugate eigenvalues) using the method of harmonic balancing.

(3) Create the main procedure *solution* for computing the normal form and non-linear transformation (i.e., periodic solution).
   — Establish ordered perturbation equations f_i by separating the original differential equations Dx_i according to the powers of ε.
   — Create the formal solutions for x_i, $i = 1, 2, 3, 4$, given in terms of basic trigonometric functions.
   — Call the procedure *Findcoef* to find the harmonic coefficients of the original differential equations after the formal solutions are used.
   — Find the normal form terms $D_n r_1, D_n p_1, D_n r_2, D_n p_2$ by eliminating the secular terms.
   — Call the procedure *solu1A_B* to find the solutions for x_i, $i = 1, 2, 3, 4$.
   — For the part associated with the non-zero real eigenvalues, call the procedure *Findcoef* first and then the procedure *solu2A_B* to obtain the solutions for x_i, $i = 5, 6, \ldots, N_1 + 4$.
   — For the part associated with the complex conjugate eigenvalues, call the procedure *Findcoef* first and then the procedure *solu3A_B* to solve the solutions for x_i, $i = N_1 + 5, N_1 + 6, \ldots, N_1 + 2 \times N_2$.

[†]The Maple source codes and sample computer input files can be found from the author's website: http://pyu1.apmaths.uwo.ca/pyu/software. The source code name is *program*5 and the name of the sample input file is *input*5.

TABLE 1

Input file to Maple source code

```
N1        :=   1:
N2        :=   1:
N         :=   4 + N1 + N2*2:
norder    :=   4:
case      :=   10:
Dx[1]     :=     omg1*x[2] + x[1]^2 + x[1]*x[3] + 2/3*x[3]*x[4] + x[4]*x[5]:
Dx[2]     := − omg1*x[1] + 2*x[4]^2 + x[5]^2 + 1/2*x[1]*x[2] + x[2]*x[4]:
Dx[3]     :=     omg2*x[4] + 4*x[3]^2 + x[5]^2 + x[1]*x[2]:
Dx[4]     := − omg2*x[3] + 2*x[2]^2 + x[3]*x[4] + x[3]*x[5]:
Dx[5]     := − x[5] + x[2]^2 + 1/3*x [2]*x[4]:
Dx[6]     := − x[6] + x[7] + x[2]^2 + 5*x[1]*x[3]:
Dx[7]     := − x[6] − x[7] + 2/5*x[1]^2:
if   case = 11 then
     omg1     := 1:
     omg2     := 2:
     Dx[1]    := Dx[1] + x[3]:
     Dx[2]    := Dx[2] + x[4]:
else
     omg1     := 2:
     omg2     := 3:
fi:
```

(4) Transform the original differential equations into the summation of ordered terms by introducing perturbation parameter ε.

(5) Transform the original variables x_i into the summation of ordered terms by introducing perturbation parameter ε.

(6) Set the zero order solutions x_{i0} and zero order normal form terms $D_0 r_1$, $D_0 p_1$, $D_0 r_2$, $D_0 p_2$.

(7) Call the main procedure *solution* to obtain the normal form and solutions for an arbitrary perturbation order n.

(8) Write the normal form $dr1$, $dp1$, $dr2$, $dp2$ into the output file "Nform".

3.2. CREATE THE INPUT FILE

(1) Set the variables:

$N1$—the number of the non-zero real eigenvalues of the Jacobian.

$N2$—the number of pairs of complex conjugate eigenvalues of the Jacobian.

N—the dimension of the system.

norder—the order of normal forms to be computed.

(2) Create the vector field, i.e., the functions Dx_i, $i = 1, 2, \ldots, n$; $n = 4 + N_1 + 2N_2$, the dimension of the system.

As an example, the input file for the third example ($2:3$ resonant case) given in the next section is shown in Table 1.

4. EXAMPLES

In this section, three examples are presented to show the applicability of the method and the efficiency of the Maple programs. The first two examples are chosen from the problems

which have been studied before, in order to verify the results obtained using the programs developed in this paper. The third example is used to demonstrate the computational efficiency of the Maple programs for computing high order normal forms of a higher-dimensional system. All the results including the normal forms and corresponding non-linear transformations are obtained by executing the Maple programs on a PC (PENTIUM III-700MMX 256K system).

4.1. EXAMPLE 1 FOR 1:2 RESONANCE

Consider the following four-dimensional general system, described by

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}), \quad \mathbf{x} \in R^4, \quad \boldsymbol{\mu} \in R^2, \tag{32}$$

where $\boldsymbol{\mu}$ is a two-dimensional vector parameter called perturbation parameter (unfolding), and the Jacobian is given by

$$J = \begin{bmatrix} 0 & \omega_{1c} & 0 & 0 \\ -\omega_{1c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{2c} \\ 0 & 0 & -\omega_{2c} & 0 \end{bmatrix}_{(\mathbf{x}=\mathbf{0},\,\boldsymbol{\mu}=\mathbf{0}),} \tag{33}$$

satisfying $\omega_{1c}/\omega_{2c} = 1:2$. Thus, we may assume that $\omega_{1c} = 1$ and $\omega_{2c} = 2$. This example was studied before using the IHB technique [16]. Since our interest here is focused on the computation of the normal form, we may ignore the perturbation parameter $\boldsymbol{\mu}$ (or simply set $\boldsymbol{\mu} = \mathbf{0}$). The general normal form up to second order was obtained using the IHB method in reference [16] as

$$\frac{\mathrm{d}r_1}{\mathrm{d}\tau_1} = r_1 r_2 (A \cos \Phi + B \sin \Phi), \qquad r_1 \frac{\mathrm{d}\phi_1}{\mathrm{d}\tau_1} = r_1 r_2 (B \cos \Phi - A \sin \Phi),$$

$$\frac{\mathrm{d}r_2}{\mathrm{d}\tau_2} = r_1^2 (C \cos \Phi + D \sin \Phi), \qquad r_2 \frac{\mathrm{d}\phi_2}{\mathrm{d}\tau_2} = r_1^2 (- D \cos \Phi + C \sin \Phi), \tag{34}$$

where $\Phi = 2\phi_1 - \phi_2$, and

$$A = \tfrac{1}{4}[(f_{113} - f_{223}) + (f_{124} + f_{214})], \qquad C = \tfrac{1}{4}[(f_{114} - f_{224}) - (f_{123} + f_{213})],$$

$$C = \tfrac{1}{4}[(f_{311} - f_{322}) + (f_{412} + f_{421})], \qquad D = \tfrac{1}{4}[(f_{411} - f_{422}) - (f_{312} + f_{321})], \tag{35}$$

in which $f_{ijk} = \partial f_i(\mathbf{0}, \mathbf{0})/\partial x_j \partial x_k$ for $\mathbf{f} = (f_1, f_2, f_3, f_4)^{\mathrm{T}}$. Note that the scaled times $\tau_2 = 2\tau_1 \equiv 2\tau$, and $\tau_k = \omega_k t$.

Now rewrite equation (32) in the expanded form

$$\dot{\mathbf{x}} = J\mathbf{x} + Q(\mathbf{x}) + \cdots, \tag{36}$$

where $Q(\mathbf{x}) = (q_1, q_2, q_3, q_4)^{\mathrm{T}}$ represents the quadratic terms with

$$q_i = \tfrac{1}{2}f_{111}\, x_1^2 + \tfrac{1}{2}f_{122}\, x_1^2 + \tfrac{1}{2}f_{133}\, x_3^2 + \tfrac{1}{2}f_{144}\, x_4^2 + f_{112}\, x_1 x_2 + f_{113} x_1 x_3$$

$$+ f_{114}\, x_1 x_4 + f_{123}\, x_2 x_3 + f_{124}\, x_2 x_4 + f_{134} x_3 x_4 \quad (i = 1, 2, 3, 4). \tag{37}$$

Then executing the Maple program developed in this paper yields the following normal form:

$$\frac{dr_1}{dt} = -r_1 r_2 (A \cos \Psi + B \sin \Psi), \qquad r_1 \frac{d\theta_1}{dt} = r_1 - r_1 r_2 (B \cos \Psi - A \sin \Psi),$$

$$\frac{dr_2}{dt} = -\frac{1}{2} r_1^2 (C \cos \Psi + D \sin \Psi), \qquad r_2 \frac{d\theta_2}{dt} = 2r_2 - \frac{1}{2} r_1^2 (-D \cos \Psi + C \sin \Psi), \qquad (38)$$

where A, B, C and D are given in equation (35), and $\Psi = \theta_2 - 2\theta_1$. One can show that equation (38) is actually identical to equation (34) by noting the different time scales, and the phase difference between system (38) and (34) (due to the different notations used in the Maple program), which is $-\pi/2$. This indicates that the results obtained in this paper using the multiple scales are the same as those given by a different method such as the IHB technique [16].

Here, it should be pointed out that the Maple program developed in this paper can be employed to find normal forms up to an arbitrary order as long as the machine memory is allowed. However, for the IHB method, only the form up to the leading order terms is correct. For example, for the 1 : 2 resonant case, it can be shown that, in general, the normal form obtained using the IHB approach is correct only up to the second order.

4.2. EXAMPLE 2 FOR 1:1 RESONANCE

The double pendulum system shown in Figure 1 consists of two rigid weigthless links of equal length l which carry two concentrated masses $2m$ and m respectively. A follower force P_1 and a constant directional force (vertical) P_2 are applied to this system. This system has been studied by many authors for different types of bifurcations (e.g., see references [14, 17]).
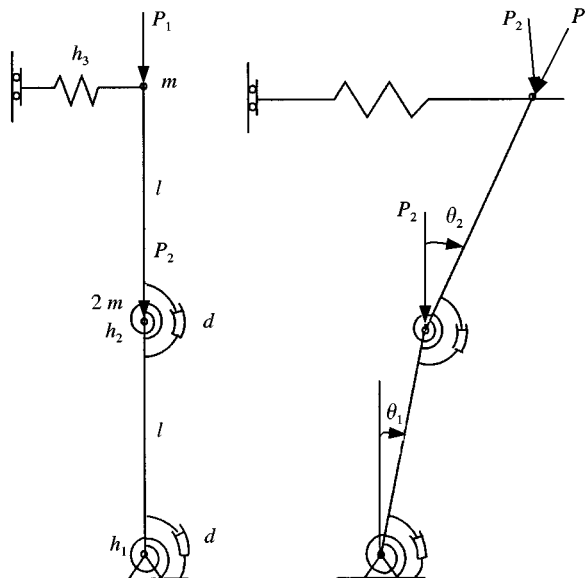

Figure 1. A double pendulum system.

The system energy for the three linear springs h_1, h_2 and h_3 is assumed to be given by [17]

$$V = \tfrac{1}{2}[(h_1 + h_2 + h_3 l^2)\theta_1^2 + 2(h_3 l^2 - h_2)\theta_1\theta_2 + (h_2 + h_3 l^2)\theta_2^2] - \tfrac{1}{6}h_3 l^2(\theta_1 + \theta_2)(\theta_1^3 + \theta_2^3),$$

(39)

where θ_1 and θ_2 are the generalized co-ordinates which specify the configuration of the system completely.

The kinetic energy T of the system is expressed by

$$T = \frac{ml^2}{2\Omega^2}[3\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)],$$

(40)

where Ω is an arbitrary value rendering the time variable non-dimensional, and the overdot denotes differentiation with respect to the non-dimensional time variable τ and $\tau = \Omega t$.

The components of the generalized forces corresponding to the generalized co-ordinates θ_1 and θ_2 may be written as

$$Q_1 = P_1 l\sin(\theta_1 - \theta_2) + 2P_2 l\sin\theta_2, \qquad Q_2 = P_2 l\sin\theta_2$$

(41)

and the damping can be expressed by

$$D = \tfrac{1}{2}[d_1\dot{\theta}_1^2 + d_2(\dot{\theta}_1 - \dot{\theta}_2)^2] + \tfrac{1}{4}[d_3\dot{\theta}_1^4 + d_4(\dot{\theta}_1 - \dot{\theta}_2)^4],$$

(42)

where d_1 and d_2 represent the linear parts, while d_3 and d_4 describe the non-linear parts.

With the aid of the *Lagrangian equations*, in addition, choosing the state variables

$$z_1 = \theta_1, \qquad z_2 = \dot{\theta}_1, \qquad z_3 = \theta_2 \quad \text{and} \quad z_4 = \dot{\theta}_2$$

(43)

and by rescaling the coefficients to be dimensionless coefficients as

$$f_1 = \frac{h_1\Omega^2}{ml^2}, \quad f_2 = \frac{h_2\Omega^2}{ml^2}, \quad f_3 = \frac{h_3\Omega^2}{m}, \quad f_4 = \frac{P_1\Omega^2}{ml}, \quad f_5 = \frac{P_2\Omega^2}{ml},$$

$$f_6 = \frac{d_3\Omega^4}{ml^2}, \quad f_7 = \frac{d_4\Omega^4}{ml^2}, \quad \eta_1 = \frac{d_1\Omega^2}{ml^2}, \quad \eta_2 = \frac{d_2\Omega^2}{ml^2},$$

(44)

one can derive a set of first order differential equations up to third order terms as follows:

$$\frac{dz_1}{d\tau} = z_2,$$

$$\begin{aligned}
\frac{dz_2}{d\tau} = {}&-(\tfrac{1}{2}f_1 + f_2 - \tfrac{1}{2}f_4 - f_5)z_1 - (\tfrac{1}{2}\eta_1 + \eta_2)z_2 + (f_2 - \tfrac{1}{2}f_4 - \tfrac{1}{2}f_5)z_3 + \eta_2 z_4 \\
&+ (\tfrac{1}{4}f_1 + \tfrac{3}{4}f_2 - \tfrac{1}{3}f_4 - \tfrac{2}{3}f_5)z_1^3 - (\tfrac{3}{4}f_2 + \tfrac{1}{2}f_3 - \tfrac{1}{3}f_4 - \tfrac{7}{12}f_5)z_3^3 + f_7 z_4^3 \\
&- (\tfrac{1}{2}f_6 + f_7)z_2^3 + (\tfrac{1}{4}\eta_1 + \tfrac{3}{4}\eta_2)z_1^2 z_2 - (\tfrac{1}{2}f_1 + \tfrac{9}{4}f_2 - \tfrac{1}{2}f_3 - f_4 - \tfrac{3}{2}f_5)z_1^2 z_3 \\
&- \tfrac{3}{4}\eta_2 z_1^2 z_4 - \tfrac{1}{2}z_1 z_2^2 + \tfrac{1}{2}z_2^2 z_3 + 3f_7 z_2^2 z_4 + (\tfrac{1}{4}f_1 + \tfrac{9}{4}f_2 - f_4 - \tfrac{3}{2}f_5)z_1 z_3^2 \\
&+ (\tfrac{1}{4}\eta_1 + \tfrac{3}{4}n_2)z_2 z_3^2 - \tfrac{3}{4}\eta_2 z_3^2 z_4 - \tfrac{1}{2}z_1 z_4^2 - 3f_7 z_2 z_4^2 + \tfrac{1}{2}z_3 z_4^2 \\
&- (\tfrac{1}{2}\eta_1 + \tfrac{3}{2}n_2)z_1 z_2 z_3 + \tfrac{3}{2}\eta_2 z_1 z_3 z_4,
\end{aligned}$$

$$\frac{\mathrm{d}z_3}{\mathrm{d}\tau} = z_4,$$

$$\frac{\mathrm{d}z_4}{\mathrm{d}\tau} = (\tfrac{1}{2}f_1 + 2f_2 - f_3 - \tfrac{1}{2}f_4 - f_5)z_1 + (\tfrac{1}{2}\eta_1 + 2\eta_2)z_2 - (2f_2 + f_3 - \tfrac{1}{2}f_4 - \tfrac{3}{2}f_5)z_3$$

$$- 2\eta_2 z_4 - (\tfrac{1}{2}f_1 + \tfrac{5}{4}f_2 - \tfrac{1}{6}f_3 - \tfrac{7}{12}f_4 - \tfrac{7}{6}f_5)z_1^3 + (\tfrac{5}{4}f_2 + \tfrac{7}{6}f_3 - \tfrac{7}{12}f_4 - f_5)z_3^3$$

$$+ (\tfrac{1}{2}f_6 + 2f_7)z_3^3 - 2f_7 z_4^3 - (\tfrac{1}{2}\eta_1 + \tfrac{5}{4}\eta_2)z_1^2 z_2 + \tfrac{5}{4}\eta_2 z_1^2 z_4 + \tfrac{3}{2}z_1 z_2^2$$

$$+ (f_1 + \tfrac{15}{4}f_2 - \tfrac{1}{2}f_3 - \tfrac{7}{4}f_4 - \tfrac{11}{4}f_5)z_1^2 z_3 - (\tfrac{1}{2}f_1 + \tfrac{15}{4}f_2 - \tfrac{1}{2}f_3 - \tfrac{7}{4}f_4 - \tfrac{5}{2}f_5)z_1 z_3^2$$

$$- \tfrac{3}{2}z_2^2 z_3 - 6f_7 z_2^2 z_4 - (\tfrac{1}{2}\eta_1 + \tfrac{5}{4}\eta_2)z_2 z_3^2 + \tfrac{5}{4}\eta_2 z_3^2 z_4 + \tfrac{1}{2}z_1 z_4^2$$

$$+ 6f_7 z_2 z_4^2 - \tfrac{1}{2}z_3 z_4^2 + (\eta_1 + \tfrac{5}{2}\eta_2)z_1 z_2 z_3 - \tfrac{5}{2}\eta_2 z_1 z_3 z_4, \tag{45}$$

where $f_i \geqslant 0$ (for $i = 1, 2, 3$) due to physical conditions, and η_1 and η_2 are used to indicate the system parameters. Note that η_1 and η_2 represent the linear parts of the damping, assumed to be non-negative.

We choose a critical point defined by

$$\eta_{1c} = \eta_{2c} = 0 \tag{46}$$

and the parameter values

$$f_1 = 10, \quad f_2 = 2, \quad f_3 = 5, \quad f_4 = -6, \quad f_5 = 8, \quad f_6 = -14, \quad f_7 = 30, \tag{47}$$

at which the Jacobian of equation (45) has non-semisimple double imaginary eigenvalues:

$$\lambda_1 = I, \quad \lambda_2 = -I, \quad \lambda_3 = I, \quad \lambda_4 = -I. \tag{48}$$

This $1{:}1$ resonant case has been studied in reference [14], where the normal form was obtained using Poincaré normal form theory. In the following, we list the two normal forms for a comparison. The Maple program developed in reference [9] based on Poincaré normal form theory can be applied to find the following normal form for system (45) up to third order, given in polar co-ordinates:

$$\dot{r}_1 = r_2 \cos(\psi), \qquad r_1 \dot{\theta}_1 = r_1 + r_2 \sin(\psi),$$

$$\dot{r}_2 = 21 r_1^2 r_2 - 5r_1^3 \cos(\psi) + \frac{21}{2} r_1^3 \sin(\psi) - 58 r_1 r_2^2 \cos(\psi) + 63 r_1 r_2^2 \sin(\psi)$$

$$- 21 r_1^2 r_2 \cos(2\psi) + r_1^2 r_2 \sin(2\psi),$$

$$r_2 \dot{\theta}_2 = r_2 - 22r_1^2 r_2 + \frac{21}{2} r_1^3 \cos(\psi) + 5 r_1^3 \sin(\psi)$$

$$+ 63 r_1 r_2^2 \cos(\psi) + 58 r_1 r_2^2 \sin(\psi) + r_1^2 r_2 \cos(2\psi) + \frac{21}{2} r_1^2 r_2 \sin(2\psi), \tag{49}$$

where the phase difference $\psi = \theta_2 - \theta_1$. Note that these four differential equations are not independent, and can be reduced to three independent equations by combining the second

and fourth equations to form an equation for $\dot{\psi}$. Note that since normal forms are generally not unique, equation (49) only represents one of the normal forms.

Similarly, we can execute the Maple program developed in this paper based on the method of multiple scales to obtain the normal form:

$$\dot{r}_1 = r_2 \cos(\psi) - \frac{21}{2} r_1^3 - \frac{105}{4} r_1 r_2^2 + \frac{83}{4} r_1^2 r_2 \cos(\psi) - \frac{315}{8} r_1^2 r_2 \sin(\psi)$$

$$+ \frac{131}{16} r_2^3 \cos(\psi) + \frac{22347}{32} r_2^3 \sin(\psi) + \frac{21}{2} r_1 r_2^2 \cos(2\psi) - 31 r_1 r_2^2 \sin(2\psi),$$

$$r_1 \dot{\theta}_1 = r_1 + r_2 \sin(\psi) - 9 r_1^3 - 12 r_1 r_2^2 - \frac{63}{8} r_1^2 r_2 \cos(\psi) + \frac{129}{4} r_1^2 r_2 \sin(\psi)$$

$$- \frac{22347}{32} r_2^3 \cos(\psi) + \frac{131}{16} r_2^3 \sin(\psi) + 31 r_1 r_2^2 \cos(2\psi) + \frac{21}{2} r_1 r_2^2 \sin(2\psi),$$

$$\dot{r}_2 = 41 r_1^2 r_2 - \frac{1971}{4} r_2^3 - 5 r_1^3 \cos(\psi) + \frac{21}{2} r_1^3 \sin(\psi)$$

$$+ \frac{21}{2} r_1 r_2^2 \cos(\psi) - \frac{53}{4} r_1 r_2^2 \sin(\psi) - \frac{21}{2} r_1^2 r_2 \cos(2\psi) + 10 r_1^2 r_2 \sin(2\psi),$$

$$r_2 \dot{\theta}_2 = r_2 - 4 r_1^2 r_2 - \frac{63}{4} r_2^3 + \frac{21}{2} r_1^3 \cos(\psi) + 5 r_1^3 \sin(\psi)$$

$$+ \frac{189}{8} r_1 r_2^2 \cos(\psi) + \frac{71}{4} r_1 r_2^2 \sin(\psi) + 10 r_1^2 r_2 \cos(2\psi) + \frac{21}{2} r_1^2 r_2 \sin(2\psi). \tag{50}$$

It is obvious from equations (49) and (50) that the normal form obtained using the multiple scales has more terms than that obtained using Poincaré normal form theory. However, the normal forms obtained using the two different methods for the non-resonance and all resonant cases (except for the 1:1 resonance) have been found to actually have the same number of terms up to a fixed order. The difference that appeared in the 1:1 resonance is due to the fact that the approach of multiple scales chooses a fixed basis and thus the form is uniquely determined at each order, while in applying Poincaré normal form theory, the form is not uniquely determined and one may select a different basis which may result in different number of terms. However, we can show that normal form (50) is equivalent to normal form (49). In other words, we can find a third order near identity non-linear transform which transforms equation (50) into equation (49). The proof is given in Appendix A. Further, it can be shown that we may use additional undetermined coefficients, the so-called *relaxing parameter* approach which has been applied to compute the simplest normal forms [18], in the process of calculating the normal form to directly find the normal form for the 1:1 resonance given in the form of equation (49). This will be discussed in more detail in a different paper.

Since the bifurcation analysis results obtained from equation (49) have been verified by a numerical approach performed based on the original system (45) [14], we will not repeat the numerical simulation here to verify equation (50).

4.3. EXAMPLE 3 FOR 2:3 RESONANCE

In order to further demonstrate the efficiency of the Maple program for computing high order normal form of higher-dimensional systems without applying center manifold theory, consider the following seven-dimensional system:

$$\dot{x}_1 = \omega_1 x_2 + x_1^2 + x_1 x_3 + \tfrac{2}{3} x_3 x_4 + x_4 x_5,$$

$$\dot{x}_2 = \omega_1 x_1 + 2x_4^2 + x_5^2 + \tfrac{1}{2} x_1 x_2 + x_2 x_4,$$

$$\dot{x}_3 = \omega_2 x_4 + 4x_3^2 + x_5^2 + x_1 x_2,$$

$$\dot{x}_4 = -\omega_2 x_3 + 2 x_2^2 + x_3 x_4 + x_3 x_5,$$

$$\dot{x}_5 = -x_5 + x_2^2 + \tfrac{1}{3} x_2 x_4,$$

$$\dot{x}_6 = -x_6 + x_7 + x_2^2 + 5 x_1 x_3,$$

$$\dot{x}_7 = -x_6 - x_7 + \tfrac{2}{5} x_1^2. \tag{51}$$

Executing the Maple program on a PC (PENTIUM III-700MMX 256K system) takes about 27 min to obtain the normal form up to 10th order. The normal form up to sixth order is given below

$$\dot{r}_1 = \frac{29}{84} r_1^3 + \frac{29}{104} r_1 r_2^2 + \frac{12713}{78336} r_1^2 r_2^2 \cos(\psi) + \frac{24143}{19584} r_1^2 r_2^2 \sin(\psi)$$

$$+ \frac{23630620163}{22368648960} r_1^5 + \frac{138177606761}{681135436800} r_1^3 r_2^2 + \frac{177068030053}{49695085440} r_1 r_2^4$$

$$- \frac{3174158149507271}{10440611639377920} r_1^4 r_2^2 \cos(\psi) + \frac{2404182693451369}{1305076454922240} r_1^4 r_2^2 \sin(\psi)$$

$$+ \frac{79586782468307866859}{55648460037884313600} r_1^4 r_2^2 \cos(\psi) - \frac{8438641581728237917}{55648460037884313600} r_1^2 r_2^4 \sin(\psi),$$

$$r_1 \dot{\theta}_1 = 2r_1 - \frac{313}{1344} r_1^3 - \frac{263}{2184} r_1 r_2^2 - \frac{24143}{19584} r_1^2 r_2^2 \cos(\psi) + \frac{12713}{78336} r_1^2 r_2^2 \sin(\psi)$$

$$- \frac{4053892009}{58432389120} r_1^5 + \frac{7248108167809}{2043406310400} r_1^3 r_2^2 + \frac{12602271904219}{12986982328320} r_1 r_2^4$$

$$- \frac{373702778230261}{522030581968896} r_1^4 r_2^2 \cos(\psi) + \frac{23211401849611067}{10440611639377920} r_1^4 r_2^2 \sin(\psi)$$

$$+ \frac{8438641581728237917}{55648460037884313600} r_1^2 r_2^4 \cos(\psi) + \frac{79586782468307866859}{55648460037884313600} r_1^2 r_2^4 \sin(\psi),$$

$$\dot{r}_2 = \frac{41}{42} r_1^2 r_2 + \frac{1258879}{1268064} r_1^3 r_2 \cos(\psi) + \frac{868757}{2536128} r_1^3 r_2 \sin(\psi)$$

$$- \frac{2000235383}{1091563200} r_1^4 r_2 + \frac{2427640143403}{1260100558080} r_1^2 r_2^3 - \frac{39909823}{49230720} r_2^5$$

$$+ \frac{629839075170792271}{278996344363376640} r_1^5 r_2 \cos(\psi) - \frac{3982767058488934319}{1673978066180259840} r_1^5 r_2 \sin(\psi)$$

$$+ \frac{1000025370085244051}{176767944831878400} r_1^3 r_2^3 \cos(\psi) - \frac{221984821019707783}{282828711731005440} r_1^3 r_2^3 \sin(\psi),$$

$$r_2 \dot{\theta}_2 = 3 r_2 - \frac{5}{84} r_1^2 r_2 - \frac{5}{8} r_2^3 + \frac{868757}{2536128} r_1^3 r_2 \cos(\psi) - \frac{1258879}{1268064} r_1^3 r_2 \sin(\psi)$$

$$+ \frac{19758721}{30321200} r_1^4 r_2 - \frac{23095337868209}{6930553069440} r_1^2 r_2^3 + \frac{8592833}{16410240} r_2^5$$

$$- \frac{3982767058488934319}{1673978066180259840} r_1^5 r_2 \cos(\psi) - \frac{629839075170792271}{278996344363376640} r_1^5 r_2 \sin(\psi)$$

$$- \frac{29817963307132463}{68743089656841600} r_1^3 r_2^3 \cos(\psi) - \frac{2692641827024565277}{1583840785693630464} r_1^3 r_2^3 \sin(\psi), \tag{52}$$

where $\psi = 2\theta_2 - 3\theta_1$.

5. CONCLUSIONS

A perturbation technique and Maple computer programs have been developed for computing the explicit normal forms of resonant double Hopf bifurcations. For a given arbitrary n-dimensional system, this approach does not require the application of center manifold theory, which has been included in the perturbation technique so that the normal form on the center manifold and the associated non-linear transformations can be obtained simultaneously. The formulas are given in an explicit iterative procedure, and thus are implemented using computer algebra. Maple programs have been developed for automating the computations. Examples are presented to verify the method and to show that the method is computationally efficient, which is particularly useful for high-dimensional systems and higher order normal forms.

REFERENCES

1. J. GUCKENHEIMER and P. HOLMES 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* New York: Springer-Verlag.
2. A. H. NAYFEH 1993 *Methods of Normal Forms.* New York: Springer-Verlag.

3. S. N. CHOW, C. LI and D. WANG 1994 *Normal Forms and Bifurcation of Planar Vector Fields.* Cambridge: Cambridge University Press.
4. J. CARR 1981 *Applications of Center Manifold Theory.* New York: Springer-Verlag.
5. R. H. RAND and W. L. KEITH 1986 in *Applications of Computer Algebra* (R. Pavelle, editor) 309–328. Boston: Kluwer Academic Press. Normal forms and center manifold calculations on MACSYMA.
6. S. N. CHOW, B. DRACHMAN and D. WANG 1991 *Journal of Computation and Applied Mathematics* **29**, 129–143. Computation of normal forms.
7. P. YU 1988 *Journal of Sound and Vibration* **211**, 19–38. Computation of normal forms via a perturbation technique.
8. Q. BI and P. YU 1998 *International Journal of Bifurcation and Chaos* **8**, 2279–2319. Computation of normal forms of differential equations associated with non-semi-simple zero eigenvalues.
9. Q. BI and P. YU 1999 *Mathematical and Computer Modelling* **29**, 49–70. Symbolic software development for computing the normal forms of double Hopf bifurcation.
10. P. YU, W. ZHANG and Q. BI 2001 *International Journal of Non-Linear Mechanics* **36**, 597–627. Vibration analysis on a thin plate with the aid of computation of normal forms.
11. W. ZHANG and K. HUSEYIN 2000 *Applied Mathematical Modelling* **24**, 279–295. On the relation between the methods of averaging and normal forms.
12. S. ZHU and P. YU 1999 *Proceedings of the 17th CANCAM, Hamilton, Canada,* 69–70, May 30–June 3. Computing normal forms for vibration analysis of nonlinear oscillators via multiple time scales.
13. P. YU *Nonlinear Dynamics* (in press). Analysis on double Hopf bifurcation using computer algebra with the aid of multiple scales.
14. P. YU and Q. BI 1998 *Journal of Sound and Vibration* **217**, 691–736. Analysis of non-linear dynamics and bifurcations of a double pendulum.
15. K. HUSEYIN 1986 *Multiple-Parameter Stability Theory and its Applications.* Oxford: Oxford University Press.
16. P. YU and K. HUSEYIN 1993 *Quarterly Applied Mathematics* **51**, 91–100. On phase-locked motions associated with strong resonance.
17. V. MANDADI and K. HUSEYIN 1980 *International Journal of Non-linear Mechanics* **15**, 159–172.
18. P. YU 2000 *International Conference on Differential Equations* (*Berlin* 1999), Vol. 2, 832–837. Singapore: World Scientific. A method for computing center manifold and normal forms.

APPENDIX A: THE PROOF FOR EXAMPLE 2

To prove that normal form (50) obtained using the multiple scales is equivalent to normal form (49) obtained using Poincaré normal form theory, we first transform the two normal forms, which are given in polar co-ordinates, into Cartesian co-ordinates. To achieve this, applying the transformation

$$x_1 = r_1 \cos(\theta_1), \qquad x_2 = r_1 \sin(\theta_1), \qquad x_3 = r_2 \cos(\theta_2), \qquad x_4 = r_2 \sin(\theta_2), \quad (A.1)$$

to equation (50) yields

$$\dot{x}_1 = x_2 + x_3 - \frac{21}{2} x_1^3 - 9 x_2^3 + \frac{131}{16} x_3^3 - \frac{22347}{32} x_4^3 - 9 x_1^2 x_2 + \frac{83}{4} x_1^2 x_3 + \frac{315}{8} x_1^2 x_4$$

$$- \frac{21}{2} x_1 x_2^2 + \frac{129}{4} x_2^2 x_3 - \frac{63}{8} x_2^2 x_4 - \frac{63}{4} x_1 x_3^2 - 43 x_2 x_3^2 - \frac{22347}{32} x_3^2 x_4 - \frac{147}{4} x_1 x_4^2$$

$$+ 19 x_2 x_4^2 + \frac{131}{16} x_3 x_4^2 - \frac{189}{4} x_1 x_2 x_3 - \frac{23}{2} x_1 x_2 x_4 + 62 x_1 x_3 x_4 + 21 x_2 x_3 x_4,$$

$$\dot{x}_2 = -x_1 + x_4 + 9x_1^3 - \frac{21}{2}x_2^3 + \frac{22347}{32}x_3^3 + \frac{131}{16}x_4^3 - \frac{21}{2}x_1^2x_2 + \frac{63}{8}x_1^2x_3 + \frac{129}{4}x_1^2x_4$$

$$+ 9x_1x_2^2 - \frac{315}{8}x_2^2x_3 + \frac{83}{4}x_2^2x_4 - 19\,x_1x_3^2 - \frac{147}{4}x_2x_3^2 + \frac{131}{16}x_3^2x_4 + 43x_1x_4^2$$

$$- \frac{63}{4}x_2x_4^2 + \frac{22347}{32}x_3x_4^2 - \frac{23}{2}x_1x_2x_3 + \frac{189}{4}x_1x_2x_4 + 21\,x_1x_3x_4 - 62x_2x_3x_4,$$

$$\dot{x}_3 = x_4 - 5x_1^3 + \frac{21}{2}x_2^3 - \frac{1971}{4}x_3^3 - \frac{63}{4}x_4^3 + \frac{21}{2}x_1^2x_2 + \frac{63}{2}x_1^2x_3 - 14x_1^2x_4$$

$$- 5x_1x_2^2 + \frac{105}{2}x_2^2x_3 + 6x_2^2x_4 - \frac{53}{4}x_1x_3^2 + \frac{735}{8}x_2x_3^2 - \frac{22347}{32}x_3^2x_4 - \frac{71}{4}x_1x_4^2$$

$$+ \frac{189}{8}x_2x_4^2 - \frac{1971}{4}x_3x_4^2 + 20\,x_1x_2x_3 - 21\,x_1x_2x_4 - \frac{273}{4}x_1\,x_3\,x_4 + \frac{9}{2}x_2x_3x_4,$$

$$\dot{x}_4 = -x_3 - \frac{21}{2}x_1^3 - 5x_2^3 + \frac{63}{4}x_3^3 - \frac{1971}{4}x_4^3 - 5x_1^2x_2 - 6x_1^2x_3 + \frac{105}{2}x_1^2x_4$$

$$- \frac{21}{2}x_1x_2^2 + 14x_2^2x_3 + \frac{63}{2}x_2^2x_4 - \frac{189}{8}x_1x_3^2 - \frac{71}{4}x_2x_3^2 - \frac{1971}{4}x_3^2x_4 - \frac{735}{8}x_1x_4^2$$

$$- \frac{53}{4}x_4x_4^2 + \frac{63}{4}x_3x_4^2 - 21x_1x_2x_3 - 20x_1x_2x_4 + \frac{9}{2}x_1x_3x_4 + \frac{273}{4}x_2x_3x_4. \tag{A.2}$$

A similar transformation can be used for equation (49) to obtain

$$\dot{y}_1 = y_2 + y_3, \qquad \dot{y}_2 = -y_1 + y_4,$$

$$\dot{y}_3 = y_4 - 5y_1^3 + \frac{21}{2}y_2^3 + \frac{21}{2}y_1^2y_2 - 23y_1^2y_4 - 5y_1y_2^2 + 42y_2^2y_3 - 21y_2^2y_4 - 58y_1y_3^2$$

$$+ 63\,y_2y_3^2 - 58y_1y_4^2 + 63y_2y_4^2 + 2y_1y_2y_3 - 42y_1y_2y_4,$$

$$\dot{y}_4 = -y_3 - \frac{21}{2}y_1^3 - 5y_2^3 - 5y_1^2y_2 + 21y_1^2y_3 + 42y_1^2y_4 - \frac{21}{2}y_1y_2^2 + 23y_2^2y_3$$

$$- 63\,y_1y_3^2 - 58y_2y_3^2 - 63y_1y_4^2 - 58y_2y_4^2 - 42y_1y_2y_3 - 2y_1y_2y_4. \tag{A.3}$$

Then, with the aid of Maple, we can find the following near identity non-linear transformation, given by

$$x_1 = y_1 + \frac{115}{8}y_1^3 + \frac{147}{16}y_2^3 + \frac{147}{16}y_1^2y_2 - \frac{1017}{4}y_1^2y_3 - \frac{483}{8}y_1^2y_4 + \frac{115}{8}y_1y_2^2 + \frac{1101}{4}y_2^2y_3 - \frac{13}{8}y_2^2y_4$$

$$+ \frac{131}{16}y_1y_3^2 + \frac{22347}{32}y_2y_3^2 + \frac{131}{16}y_1y_4^2 + \frac{22347}{32}y_2y_4^2 + \frac{235}{4}y_1y_2y_3 - \frac{1059}{2}y_1y_2y_4,$$

$$x_2 = y_2 - \frac{147}{16}y_1^3 + \frac{115}{8}y_2^3 + \frac{115}{8}y_1^2 y_2 - \frac{13}{8}y_1^2 y_3 - \frac{1101}{4}y_1^2 y_4 - \frac{147}{16}y_1 y_2^2 + \frac{483}{8}y_2^2 y_3 - \frac{1017}{4}y_2^2 y_4$$

$$- \frac{22347}{32}y_1 y_3^2 + \frac{131}{16}y_2 y_3^2 - \frac{22347}{32}y_1 y_4^2 + \frac{131}{16}y_2 y_4^2 - \frac{1059}{2}y_1 y_2 y_3 - \frac{235}{4}y_1 y_2 y_4,$$

$$x_3 = y_3 + \frac{21}{2}y_1^3 - 9y_2^3 - 9y_1^2 y_2 + \frac{179}{8}y_1^2 y_3 + \frac{777}{16}y_1^2 y_4 + \frac{21}{2}y_1 y_2^2 - \frac{143}{8}y_2^2 y_3 + \frac{315}{16}y_2^2 y_4$$

$$- \frac{1971}{4}y_1 y_3^2 + \frac{63}{4}y_2 y_3^2 - \frac{1971}{4}y_1 y_4^2 + \frac{63}{4}y_2 y_4^2 - \frac{231}{8}y_1 y_2 y_3 + \frac{161}{4}y_1 y_2 y_4,$$

$$x_4 = y_4 + 9y_1^3 + \frac{21}{2}y_2^3 + \frac{21}{2}y_1^2 y_2 - \frac{315}{16}y_1^2 y_3 - \frac{143}{8}y_1^2 y_4 + 9y_1 y_2^2 - \frac{777}{16}y_2^2 y_3 + \frac{179}{8}y_2^2 y_4$$

$$- \frac{63}{4}y_1 y_3^2 - \frac{1971}{4}y_2 y_3^2 - \frac{63}{4}y_1 y_4^2 - \frac{1971}{4}y_2 y_4^2 + \frac{161}{4}y_1 y_2 y_3 + \frac{231}{8}y_1 y_2 y_4, \qquad \text{(A.4)}$$

to reduce equations (A.2) to equations (A.3). This proves that equations (A.2) and equations (A.3) (i.e., equations (49) and equations (50)) are indeed equivalent.